Note on Elementary Analysis II (2019-20)

4. Power series

Throughout this section, let

$$f(x)=\sum_{i=0}^{\infty}a_ix^i \qquad \cdots\cdots\cdots\cdots (*)$$

denote a formal power series, where $a_i\in\mathbb{R}$.

Lemma 4.1. *Suppose that there is $c\in\mathbb{R}$ with $c\neq 0$ such that $f(c)$ is convergent. Then*

(i) : $f(x)$ is absolutely convergent for all x with $|x|<|c|$.

(ii) : f converges uniformly on $[-\eta,\eta]$ for any $0<\eta<|c|$.

Proof. For Part (i), note that since $f(c)$ is convergent, then $\lim a_nc^n=0$. So there is a positive integer N such that $|a_nc^n|\leq 1$ for all $n\geq N$. Now if we fix $|x|<|c|$, then $|x/c|<1$. Therefore, we have

$$\sum_{n=1}^{\infty}|a_n||x^n|\leq\sum_{n=1}^{N-1}|a_n||x^n|+\sum_{n\geq N}|a_nc^n||x/c|^n\leq\sum_{n=1}^{N-1}|a_n||x^n|+\sum_{n\geq N}|x/c|^n<\infty.$$

So Part (i) follows.
Now for Part (ii), if we fix $0<\eta<|c|$,then $|a_nx^n|\leq|a_n\eta|^n$ for all n and for all $x\in[-\eta,\eta]$. On the other hand, we have $\sum_n|a_n\eta^n|<\infty$ by Part (i). So f converges uniformly on $[-\eta,\eta]$ by the M-test. The proof is finished. $\square$

Remark 4.2. *In Lemma 4.9(ii), notice that if $f(c)$ is convergent, it does not imply f converges uniformly on $[-c,c]$ in general.*
For example, $f(x):=1+\sum_{n=1}^{\infty}\frac{x^n}{n}$. Then $f(-1)$ is convergent but $f(1)$ is divergent.

Definition 4.3. *Call the set dom $f:=\{x\in\mathbb{R}:f(c)\ \text{ is convergent }\}$ the domain of convergence of f for convenience. Let $0\leq r:=\sup\{|c|:c\in \text{dom } f\}\leq\infty$. Then r is called the radius of convergence of f.*

Remark 4.4. *Notice that by Lemma 4.9, then the domain of convergence of f must be the interval with the end points $\pm r$ if $0<r<\infty$.*
When $r=0$, then dom $f=\{0\}$.
Finally, if $r=\infty$, then dom $f=\mathbb{R}$.

Example 4.5. *If $f(x)=\sum_{n=0}^{\infty}n!x^n$, then $r=(0)$. In fact, notice that if we fix a non-zero number x and consider $\lim_n|(n+1)!x^{n+1}|/|n!x^n|=\infty$, then by the ratio test $f(x)$ must be divergent for any $x\neq 0$. So $r=0$ and dom $f=(0)$.*

Example 4.6. *Let $f(x)=1+\sum_{n=1}^{\infty}x^n/n^n$. Notice that we have $\lim_n|x^n/n^n|^{1/n}=0$ for all x. So the root test implies that $f(x)$ is convergent for all x and then $r=\infty$ and dom $f=\mathbb{R}$.*

Example 4.7. *Let $f(x)=1+\sum_{n=1}^{\infty}x^n/n$. Then $\lim_n|x^{n+1}/(n+1)|\cdot|n/x^n|=|x|$ for all $x\neq 0$. So by the ration test, we see that if $|x|<1$, then $f(x)$ is convergent and if $|x|>1$, then $f(x)$ is divergent. So $r=1$. Also, it is known that $f(1)$ is divergent but $f(-1)$ is divergent. Therefore, we have dom $f=[-1,1)$.*

Example 4.8. *Let $f(x) = \sum x^n/n^2$. Then by using the same argument of Example 4.7, we have $r = 1$. On the other hand, it is known that $f(\pm 1)$ both are convergent. So dom $f = [-1, 1]$.*

Lemma 4.9. *With the notation as above, if $r > 0$, then f converges uniformly on $(-\eta, \eta)$ for any $0 < \eta < r$.*

Proof. It follows from Lemma 4.1 at once. □

Remark 4.10. *Note that the Example 4.7 shows us that f may not converge uniformly on $(-r, r)$. In fact let f be defined as in Example 4.7. Then f does not converges on $(-1, 1)$. In fact, if we let $s_n(x) = \sum_{k=0}^{\infty} a_k x^k$, then for any positive integer n and $0 < x < 1$, we have*

$$|s_{2n}(x) - s_n(x)| = \frac{x^{n+1}}{n+1} + \cdots\cdots + \frac{x^n}{2n}.$$

From this we see that if n is fixed, then $|s_{2n}(x) - s_n(x)| \to 1/2$ as $x \to 1-$. So for each n, we can find $0 < x < 1$ such that $|s_{2n}(x) - s_n(x)| > \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. Thus f does not converges uniformly on $(-1, 1)$ by the Cauchy Theorem.

Proposition 4.11. *With the notation as above, let $\ell = \overline{\lim}\, |a_n|^{1/n}$ or $\lim \frac{|a_{n+1}|}{|a_n|}$ provided it exists. Then*

$$r = \begin{cases} \frac{1}{\ell} & \text{if } 0 < \ell < \infty; \\ 0 & \text{if } \ell = \infty; \\ \infty & \text{if } \ell = 0. \end{cases}$$

Proposition 4.12. *With the notation as above if $0 < r \leq \infty$, then $f \in C^\infty(-r, r)$. Moreover, the k-derivatives $f^{(k)}(x) = \sum_{n \geq k} a_k n(n-1)(n-2)\cdots\cdots(n-k+1)x^{n-k}$ for all $x \in (-r, r)$.*

Proof. Fix $c \in (-r, r)$. By Lemma 4.9, one can choose $0 < \eta < r$ such that $c \in (-\eta, \eta)$ and f converges uniformly on $(-\eta, \eta)$.
It needs to show that the k-derivatives $f^{(k)}(c)$ exists for all $k \geq 0$. Consider the case $k = 1$ first.
If we consider the series $\sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, then it also has the same radius r because $\lim_n |n a_n|^{1/n} = \lim_n |a_n|^{1/n}$. This implies that the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges uniformly on $(-\eta, \eta)$. Therefore, the restriction $f|(-\eta, \eta)$ is differentiable. In particular, $f'(c)$ exists and $f'(c) = \sum_{n=1}^{\infty} n a_n c^{n-1}$.
So the result can be shown inductively on k. □

Proposition 4.13. *With the notation as above, suppose that $r > 0$. Then we have*

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = \sum_{0}^{\infty} \frac{1}{n+1} a_n x^{n+1}$$

for all $x \in (-r, r)$.

Proof. Fix $0 < x < r$. Then by Lemma 4.9 f converges uniformly on $[0, x]$. Since each term $a_n t^n$ is continuous, the result follows. □

Theorem 4.14. **(Abel)** : *With the notation as above, suppose that $0 < r$ and $f(r)$ (or $f(-r)$) exists. Then f is continuous at $x = r$ (resp. $x = -r$), that is $\lim_{x \to r-} f(x) = f(r)$.*

Proof. Note that by considering $f(-x)$, it suffices to show that the case $x = r$ holds.
Assume $r = 1$.
Notice that if f converges uniformly on $[0, 1]$, then f is continuous at $x = 1$ as desired.
Let $\varepsilon > 0$. Since $f(1)$ is convergent, then there is a positive integer such that

$$|a_{n+1} + \cdots\cdots\cdots + a_{n+p}| < \varepsilon$$

for $n \geq N$ and for all $p = 1, 2....$ Note that for $n \geq N$; $p = 1, 2...$ and $x \in [0, 1]$, we have

$$\begin{aligned} s_{n+p}(x) - s_n(x) &= a_{n+1}x^{n+1} + a_{n+2}x^{n+1} + a_{n+3}x^{n+1} + \cdots\cdots\cdots + a_{n+p}x^{n+1} \\ &+ a_{n+2}(x^{n+2} - x^{n+1}) + a_{n+3}(x^{n+2} - x^{n+1}) + \cdots\cdots\cdots + a_{n+p}(x^{n+2} - x^{n+1}) \\ &+ a_{n+3}(x^{n+3} - x^{n+2}) + \cdots\cdots\cdots + a_{n+p}(x^{n+3} - x^{n+2}) \\ &\vdots \\ &+ a_{n+p}(x^{n+p} - x^{n+p-1}). \end{aligned} \tag{4.1}$$

Since $x \in [0, 1]$, $|x^{n+k+1} - x^{n+k}| = x^{n+k} - x^{n+k+1}$. So the Eq.4.1 implies that
$|s_{n+p}(x)-s_n(x)| \leq \varepsilon(x_{n+1}+(x^{n+1}-x^{n+2})+(x^{n+2}-x^{n+3})+\cdots+(x^{n+p-1}-x^{n+p})) = \varepsilon(2x^{n+1}-x^{n+p}) \leq 2\varepsilon.$
So f converges uniformly on $[0, 1]$ as desired.
Finally for the general case, we consider $g(x) := f(rx) = \sum_n a_n r^n x^n$. Note that $\lim_n |a_n r^n|^{1/n} = 1$ and $g(1) = f(r)$. Then by the case above,, we have shown that

$$f(r) = g(1) = \lim_{x \to 1-} g(x) = \lim_{x \to r-} f(x).$$

The proof is finished. □

Remark 4.15. *In Remark 4.10, we have seen that f may not converges uniformly on $(-r, r)$. However, in the proof of Abel's Theorem above, we have shown that if $f(\pm r)$ both exist, then f converges uniformly on $[-r, r]$ in this case.*

5. Real analytic functions

Proposition 5.1. *Let $f \in C^\infty(a, b)$ and $c \in (a, b)$. Then for any $x \in (a, b) \setminus \{c\}$ and for any $n \in \mathbb{N}$, there is $\xi = \xi(x, n)$ between c and x such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k + \int_c^x \frac{f^{(n+1)}(t)}{n!}(x - t)^n dt$$

Call $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k$ (may not be convergent) the Taylor series of f at c.

Proof. It is easy to prove by induction on n and the integration by part. □

Definition 5.2. *A real-valued function f defined on (a, b) is said to be real analytic if for each $c \in (a, b)$, one can find $\delta > 0$ and a power series $\sum_{k=0}^{\infty} a_k(x - c)^k$ such that*

$$f(x) = \sum_{k=0}^{\infty} a_k(x - c)^k \qquad \cdots\cdots\cdots (*)$$

for all $x \in (c - \delta, c + \delta) \subseteq (a, b)$.

Remark 5.3.

(i) : Concerning about the definition of a real analytic function f, the expression $(*)$ above is uniquely determined by f, that is, each coefficient a_k's is uniquely determined by f. In fact, by Proposition 4.12, we have seen that $f \in C^\infty(a,b)$ and

$$a_k = \frac{f^{(k)}(c)}{k!} \qquad \cdots\cdots\cdots (**)$$

for all $k = 0, 1, 2, ...$

(ii) : Although every real analytic function is C^∞, the following example shows that the converse does not hold.
Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

One can directly check that $f \in C^\infty(\mathbb{R})$ and $f^{(k)}(0) = 0$ for all $k = 0, 1, 2....$ So if f is real analytic, then there is $\delta > 0$ such that $a_k = 0$ for all k by the Eq.$(**)$ above and hence $f(x) \equiv 0$ for all $x \in (-\delta, \delta)$. It is absurd.

(iii) **Interesting Fact** *: Let D be an open disc in $\mathbb{C}$. A complex analytic function f on D is similarly defined as in the real case. However, we always have: f is complex analytic if and only if it is C^∞.*

Proposition 5.4. *Suppose that $f(x) := \sum_{k=0}^{\infty} a_k (x-c)^k$ is convergent on some open interval I centered at c, that is $I = (c-r, c+r)$ for some $r > 0$. Then f is analytic on I.*

Proof. We first note that $f \in C^\infty(I)$. By considering the translation $x - c$, we may assume that $c = 0$. Now fix $z \in I$. Now choose $\delta > 0$ such that $(z-\delta, z+\delta) \subseteq I$. We are going to show that

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z)}{j!}(x-z)^j.$$

for all $x \in (z-\delta, z+\delta)$.
Notice that $f(x)$ is absolutely convergent on I. This implies that

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_k (x - z + z)^k \\ &= \sum_{k=0}^{\infty} a_k \sum_{j=0}^{k} \frac{k(k-1)\cdots\cdots(k-j+1)}{j!}(x-z)^j z^{k-j} \\ &= \sum_{j=0}^{\infty} \Big(\sum_{k\geq j} k(k-1)\cdots\cdots(k-j+1) a_k z^{k-j}\Big) \frac{(x-z)^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{f^{(j)}(z)}{j!}(x-z)^j \end{aligned}$$

for all $x \in (z-\delta, z+\delta)$. The proof is finished. □

Example 5.5. *Let $\alpha \in \mathbb{R}$. Recall that $(1+x)^\alpha$ is defined by $e^{\alpha \ln(1+x)}$ for $x > -1$.*
Now for each $k \in \mathbb{N}$, put

$$\binom{\alpha}{k} = \begin{cases} \frac{\alpha(\alpha-1)\cdots\cdots(\alpha-k+1)}{k!} & \text{if } k \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$$

Then

$$f(x) := (1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

whenever $|x| < 1$.
Consequently, $f(x)$ *is analytic on* $(-1, 1)$.

Proof. Notice that $f^{(k)}(x) = \alpha(\alpha-1)\cdot \cdots \cdot(\alpha-k+1)(1+x)^{\alpha-k}$ for $|x| < 1$.
Fix $|x| < 1$. Then by Proposition 5.1, for each positive integer n we have

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \int_0^x \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} dt$$

So by the mean value theorem for integrals, for each positive integer n, there is ξ_n between 0 and x such that

$$\int_0^x \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} dt = \frac{f^{(n)}(\xi_n)}{(n-1)!}(x-\xi_n)^{n-1} x$$

Now write $\xi_n = \eta_n x$ for some $0 < \eta_n < 1$ and $R_n(x) := \dfrac{f^{(n)}(\xi_n)}{(n-1)!}(x-\xi_n)^{n-1}x$. Then

$$R_n(x) = (\alpha-n+1)\binom{\alpha}{n-1}(1+\eta_n x)^{\alpha-n}(x-\eta_n x)^{n-1}x = (\alpha-n+1)\binom{\alpha}{n-1}x^n(1+\eta_n x)^{\alpha-1}(\frac{1-\eta_n}{1+\eta_n x})^{n-1}.$$

We need to show that $R_n(x) \to 0$ as $n \to \infty$, that is the Taylor series of f centered at 0 converges to f. By the Ratio Test, it is easy to see that the series $\sum\limits_{k=0}^{\infty}(\alpha-k+1)\binom{\alpha}{k}y^k$ is convergent as $|y| < 1$. This tells us that $\lim\limits_n |(\alpha-n+1)\binom{\alpha}{n}x^n| = 0$.
On the other hand, note that we always have $0 < 1-\eta_n < 1+\eta_n x$ for all n because $x > -1$. Thus, we can now conclude that $R_n(x) \to 0$ as $|x| < 1$. The proof is finished. Finally the last assertion follows from Proposition 5.4 at once. The proof is complete. □

References

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